On Some Operators Involving Hadamard Derivatives

Roberto Garra$_1$, Federico Polito$_2$

(1) – Dipartimento di Scienze di Base ed Applicate per l'Ingegneria,

"Sapienza" Università di Roma.

Via A. Scarpa 16, 00161, Rome, Italy.

Email address: roberto.garra@sbai.uniroma1.it

(2) – Dipartimento di Matematica, Università di Torino.

Via Carlo Alberto 10, 10123, Torino, Italy.

Tel: +39-011-6702937, fax: +39-011-6702878

Email address: federico.polito@unito.it

April 4, 2013

Abstract

In this paper we introduce a novel Mittag–Leffler-type function and study its properties in relation to some integro-differential operators involving Hadamard fractional derivatives or Hyper-Bessel-type operators. We discuss then the utility of these results to solve some integro-differential equations involving these operators by means of operational methods. We show the advantage of our approach through some examples. Among these, an application to a modified Lamb–Bateman integral equation is presented.

Keywords: Hadamard derivatives; α-Mittag–Leffler functions; αL-exponential functions; Lamb–Bateman equation; Hyper-Bessel operators.

1 Introduction

Fractional calculus is a developing field of the applied analysis concerning methods and applications of integro-differential equations involving fractional operators. In literature there are many different definitions of fractional derivatives and integrals [14]. One of these definitions was introduced by Hadamard [7] in 1892 and, although not so frequently used in applications, it is generally mentioned in classical reference books on fractional calculus [14, Section 18.3]. In some recent papers by Babusci et al. [2, 1], it has been shown by means of operational techniques that the Hadamard integral corresponds to a negative fractional power of the operator $x\frac{\mathrm{d}}{\mathrm{d}x}$. By exploiting the integral representation of such operator, they have shown that the Lamb–Bateman integral equation can be rearranged as a fractional integro-differential equation involving Hadamard fractional derivatives of order 1/2. However, a very few works on applications of Hadamard integrals and derivatives have been carried out, mainly due to the intrinsic intractability of such operators. On the other hand, many papers on the properties of Laguerre derivatives and their applications in partial differential equations (see for example Dattoli et al. [3]) are present in literature. By means of operational methods Dattoli and Ricci [4] have found a simple way to solve analitically a class of generalized evolution problems involving Laguerre-type operators. The purpose of this paper is to extend these results and to consider some applications in order to solve integro-differential equations involving Hadamard fractional integrals and derivatives.
First, we define a general class of functions related to fractional operators involving Hadamard derivatives and fractional Hyper-Bessel-type operators. Some of the main properties of this class of functions and of the related operators will bee analyzed and discussed. Then we will show the advantage of this approach to solve some integro-differential equations and present a concrete example treating a modified Lamb–Bateman equation.

2 Basics concepts on Hadamard fractional calculus and Laguerre derivatives

In order to make the paper self-contained, here we recall some definitions and properties of Hadamard fractional integration and differentiation. We refer to Kilbas [9] for a more detailed analysis. The first step is to define the Hadamard fractional integral and derivative operators.

Definition 2.1. *Let $\Re(\alpha) > 0$. The Hadamard fractional integral of order α, applied to the function $f \in L^p[a,b]$, $0 \le a < b \le \infty$, $x \in [a,b]$, is defined as*

$$J^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \left(\log \frac{x}{t}\right)^{\alpha-1} f(t) \frac{\mathrm{d}t}{t}. \tag{2.1}$$

Before constructing the corresponding derivative operator, we define the following space of functions.

Definition 2.2. *Let $[a,b]$ be a finite interval such that $-\infty < a < b < \infty$ and let $AC[a,b]$ be the space of absolutely continuous functions on $[a,b]$. Let us denote $\delta = x\frac{\mathrm{d}}{\mathrm{d}x}$ and define the space*

$$AC^n_\delta[a,b] = \left\{g \colon [a,b] \to \mathbb{C} \colon \delta^{n-1}\left[g(x)\right] \in AC[a,b]\right\}. \tag{2.2}$$

Clearly $AC^1_\delta[a,b] \equiv AC[a,b]$. Analogously to the Riemann–Liouville fractional calculus, the Hadamard fractional derivative is defined in terms of the Hadamard fractional integral in the following way.

Definition 2.3. *Let $\delta = x\frac{\mathrm{d}}{\mathrm{d}x}$, $\Re(\alpha) > 0$ and $n = [\alpha]+1$, where $[\alpha]$ is the integer part of α. The Hadamard fractional derivative of order α applied to the function $f \in AC^n_\delta[a,b]$, $0 \le a < b < \infty$, is defined as*

$$D^\alpha f(x) = \delta^n \left(J^{n-\alpha} f\right)(x). \tag{2.3}$$

A more general Hadamard-type fractional operator has been studied by Kilbas [9]. We will stick anyway to the classical definition.
It has been proved (see e.g. Kilbas [9, Theorem 4.8]) that in the space $L^p[a,b]$, $0 < a < b < \infty$, $1 \le p \le \infty$, the Hadamard fractional derivative is the left-inverse operator to the Hadamard fractional integral, i.e.

$$D^\alpha J^\alpha f(x) = f(x). \tag{2.4}$$

Furthermore, we recall some recent results of Babusci et al. [1] on the operator $\left(x\frac{d}{dx}\right)^{-\alpha}$, with $\alpha > 0$. The authors have proved by means of operational techniques that it corresponds to the Hadamard fractional integral as it is shown in the following. Babusci et al. [1] have considered the following integral representation of the power $\alpha > 0$ of a general operator a

$$a^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}s\, e^{-sa} s^{\alpha-1}, \tag{2.5}$$

so that

$$\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}s\, e^{-sx\frac{\mathrm{d}}{\mathrm{d}x}} s^{\alpha-1}. \tag{2.6}$$

To explain it clearly we recall the properties of the dilation operator $e^{\lambda x \frac{\mathrm{d}}{\mathrm{d}x}}$ [13]. By considering $x = e^\theta$ we have

$$e^{\lambda x \frac{\mathrm{d}}{\mathrm{d}x}} f(x) = e^{\lambda \frac{\mathrm{d}}{\mathrm{d}\theta}} f(e^\theta) = f(e^{\theta+\lambda}) = f(e^\lambda x), \tag{2.7}$$

where we used the shift action of the exponential operator, that is $e^{\lambda\frac{\mathrm{d}}{\mathrm{d}\theta}} f(\theta) = f(\theta+\lambda)$. Going back to (2.6)

$$\begin{aligned} \left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^{-\alpha} f(x) &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}s\, e^{-sx\frac{\mathrm{d}}{\mathrm{d}x}} f(x) s^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^\infty \mathrm{d}s\, f(e^{-s}x) s^{\alpha-1} \\ &= \frac{1}{\Gamma(\alpha)} \int_0^x \left(\log\frac{x}{t}\right)^{\alpha-1} f(t)\frac{\mathrm{d}t}{t} = J^\alpha f(x). \end{aligned} \tag{2.8}$$

Since (2.4) holds we immediately have that $\left(x\frac{\mathrm{d}}{\mathrm{d}x}\right)^\alpha \equiv D^\alpha$. This is consistent with the interpretation of the Hadamard fractional derivative as a fractional power of the operator δ [14, Section 18.3].
These considerations, that will be clear in the following , imply for example that the Lamb–Bateman equation (see Section 4) can be restated as an equation involving a Hadamard fractional derivative.
We now recall the definition of the Laguerre derivative (see e.g. Dattoli et al. [3], Dattoli and Ricci [4]) as the operator $D_L = \frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\delta$. Furthermore, let us denote with $D_{nL} = \frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}\dots\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}$, $n \in \mathbb{N}\cup\{0\}$, the Laguerre operator of order n and containing $n+1$ derivatives. For example $D_{0L} \equiv \frac{\mathrm{d}}{\mathrm{d}x}$, $D_{1L} \equiv D_L$, and $D_{2L} = \frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}$. For the subsequent developments we also recall the following theorem by Dattoli and Ricci [4, Theorem 2.2] concerning the eigenfunctions of the Laguerre operator of order n.

Theorem 2.1. *The L-exponential function of order n*

$$e_n(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!^{n+1}}, \qquad n \in \mathbb{N} \cup \{0\},\ x \in \mathbb{R}, \tag{2.9}$$

is an eigenfunction of the operator D_{nL}, *i.e.* $D_{nL} e_n(ax) = a\, e_n(ax)$, $a \in \mathbb{R}$.

The L-exponential function is also called in literature as nL-exponentials [13] and includes the two relevant specific cases of the classical exponential function ($n = 0$) and the 0th-order Bessel–Tricomi function ($n = 1$).

Definition 2.4. *Let* $\alpha \in (0, \infty)$, $x > 0$, *we define the following operator of order* α.

$$\mathfrak{D}^\alpha f(x) = \frac{\mathrm{d}}{\mathrm{d}x} D^\alpha f(x). \tag{2.10}$$

Remark 2.1. *Notice that when* $\alpha \in (0, 1)$, *the operator* (2.10) *can be written as*

$$\mathfrak{D}^\alpha f(x) = \frac{\mathrm{d}}{\mathrm{d}x} \left(x \frac{\mathrm{d}}{\mathrm{d}x} \right) \left(J^{1-\alpha} f \right)(x) = D_L \left(J^{1-\alpha} f \right)(x), \tag{2.11}$$

and thus it can be seen as a generalization of Hadamard derivatives by means of Laguerre derivatives.

In the following section we will show some advantages of this approach in order to solve explicitly a class of integro-differential equations involving Hadamard and Laguerre type derivatives.

3 Equations involving the operator $\mathfrak{D}^\alpha$

We first introduce a rather general function which will be specialized in several ways.

Definition 3.1 (α-Mittag–Leffler function)**.** *The* α*-Mittag–Leffler function* $E_{\alpha;\nu,\gamma}(x)$, $x \in \mathbb{R}$, $\alpha > -1$, $\nu > 0$, $\gamma \in \mathbb{R}$, *is defined as*

$$E_{\alpha;\nu,\gamma}(x) = \sum_{k=0}^{\infty} \frac{x^k}{\Gamma^{\alpha+1}(\nu k + \gamma)}. \tag{3.1}$$

Clearly, for $\alpha = 0$ the classical Mittag–Leffler function $E_{0;\nu,\gamma}(x) = E_{\nu,\gamma}(x)$ is retrieved. The following theorem presents the associated Laplace transform for $\alpha \in (0, \infty)$, $x \geq 0$.

Theorem 3.1. *We have that*

$$\int_0^\infty e^{-sx} x^{\gamma-1} E_{\alpha;\nu,\gamma}(\lambda x^\nu)\, \mathrm{d}x = \frac{1}{s^\gamma} E_{\alpha-1;\nu,\gamma}\left(\frac{\lambda}{s^\nu} \right), \qquad s > 0,\ \lambda \in \mathbb{R},\ x \geq 0,\ \alpha \in (0, \infty). \tag{3.2}$$

Proof. By direct calculation we obtain

$$\begin{aligned} \int_0^\infty e^{-sx} x^{\gamma-1} E_{\alpha;\nu,\gamma}(\lambda x^\nu)\, \mathrm{d}x &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma^{\alpha+1}(\nu k + \gamma)} \int_0^\infty e^{-sx} x^{\nu k + \gamma - 1} \mathrm{d}x \\ &= \sum_{k=0}^{\infty} \frac{\lambda^k}{\Gamma^{\alpha+1}(\nu k + \gamma)} \frac{\Gamma(\nu k + \gamma)}{s^{\nu k + \gamma}} = \frac{1}{s^\gamma} E_{\alpha-1;\nu,\gamma}\left(\frac{\lambda}{s^\nu} \right), \end{aligned} \tag{3.3}$$

where the inversion of the sum with the integral is permitted by Theorem 30.1 of Doetsch [5]. □

Note that when $\alpha \to 0$ we obtain, as expected, the Laplace transform of the two-parameters Mittag–Leffler function, i.e.

$$\int_0^\infty e^{-sx} x^{\gamma-1} E_{\nu,\gamma}(\lambda x^\nu)\, \mathrm{d}x = \frac{s^{\nu-\gamma}}{s^\nu - \lambda}, \qquad s > |\lambda|^{1/\nu}. \tag{3.4}$$

Theorem 3.2. *For* $\alpha = n \in \mathbb{N} \cup \{0\}$, *the n-Mittag–Leffler function* $E_{n;\nu,1}(\lambda x^\nu)$, $\lambda \in \mathbb{R}$, $x \geq 0$, $\nu > 0$, *is an eigenfunction of a fractional hyper-Bessel-type operator*

$$\underbrace{\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} x^\nu \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} \cdots \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu} x^\nu \frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}}_{n+1 \text{ derivatives}}, \tag{3.5}$$

where d^ν / dx^ν *represents in this case the Caputo fractional derivative [12].*

Proof. The statement is readily proved by simply observing that

$$
\begin{aligned}
&\underbrace{\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}x^\nu\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\cdots\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}x^\nu\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}}_{n+1 \text{ derivatives}}\sum_{k=0}^\infty \frac{\lambda^k x^{\nu k}}{\Gamma^{n+1}(\nu k+1)} \qquad (3.6)\\
&=\underbrace{\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}x^\nu\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\cdots\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}x^\nu\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}}_{n \text{ derivatives}}\sum_{k=1}^\infty \frac{\lambda^k x^{\nu k}}{\Gamma^{n+1}(\nu k+1)}\frac{\Gamma(\nu k+1)}{\Gamma(\nu k+1-\nu)}\\
&=\cdots=\frac{\mathrm{d}^\nu}{\mathrm{d}x^\nu}\sum_{k=1}^\infty \frac{\lambda^k x^{\nu k}}{\Gamma^{n+1}(\nu k+1)}\frac{\Gamma^n(\nu k+1)}{\Gamma^n(\nu k+1-\nu)}\\
&=\sum_{k=1}^\infty \frac{\lambda^k x^{\nu k-\nu}}{\Gamma^{n+1}(\nu k+1-\nu)}=\lambda E_{n;\nu,1}(\lambda x^\nu), \qquad \lambda\in\mathbb{R},\ x\geq 0.
\end{aligned}
$$

□

Remark 3.1. *Notice that if $\alpha = n \in \mathbb{N}\cup\{0\}$ we can write the α-Mittag–Leffler function* (3.1) *as a generalized Wright function. Indeed*

$$
E_{n;\nu,\gamma}(x)=\sum_{k=0}^\infty \frac{x^k}{\Gamma^{n+1}(\nu k+\gamma)}={}_1\psi_{n+1}\left[x\,\middle|\,\begin{matrix}(1,1)\\ \underbrace{(\gamma,\nu),\dots,(\gamma,\nu)}_{n+1 \textit{ times}}\end{matrix}\right], \qquad x\in\mathbb{R},\ \nu>0,\ \gamma\in\mathbb{R}, \tag{3.7}
$$

where

$$
{}_p\psi_q\left[z\,\middle|\,\begin{matrix}(a_1,A_1),\dots,(a_p,A_p)\\(b_1,B_1),\dots,(b_q,B_q)\end{matrix}\right]=\sum_{k=0}^\infty\frac{\prod_{j=1}^p\Gamma(a_j+A_jk)}{\prod_{j=1}^q\Gamma(b_j+B_jk)}\frac{z^k}{k!}, \tag{3.8}
$$

with $a_j, b_j\in\mathbb{C}$, $A_j,B_j\in(0,\infty)$, $z\in\mathbb{C}$, is the generalized Wright function.
Clearly, the n-Mittag–Leffler function admits other representations, for example as a multi-index Mittag–Leffler function [10].

Remark 3.2. *It is worth noticing that for $(\nu,\gamma)=(1,1)$ the operator* (3.5) *and the function* (3.1) *coincide respectively with the operator D_{nL} and the L-exponential function* (2.9).

By means of the following definition we specialize function (3.1) for $(\nu,\gamma)=(1,1)$ thus obtaining a generalization of the L-exponential function (2.9) for fractional values of parameter n.

Definition 3.2. *The αL-exponential function $\mathfrak{e}_\alpha(x)$, $\alpha\in(-1,\infty)$, $x\in\mathbb{R}$, is defined as*

$$
\mathfrak{e}_\alpha(x)=\sum_{k=0}^\infty\frac{x^k}{k!^{\alpha+1}}. \tag{3.9}
$$

For the Cauchy–Hadamard theorem, the αL-exponential function is clearly convergent in $\mathbb{R}$ for $\alpha\in(-1,\infty)$. Its Laplace transform for $x\geq 0$, $\alpha\in(0,\infty)$ reads

$$
\int_0^\infty e^{-sx}\mathfrak{e}_\alpha(x)\,\mathrm{d}x=\frac{1}{s}\sum_{k=0}^\infty\frac{s^{-k}}{k!^\alpha}=\frac{1}{s}\,\mathfrak{e}_{\alpha-1}\left(\frac{1}{s}\right), \qquad s>0, \tag{3.10}
$$

where the inversion of the sum with the integral is permitted by Theorem 30.1 of Doetsch [5]. Note that formula (3.10) for $\alpha\to 0$, $s>1$, correctly gives the Laplace transform of the classical exponential function. Note that, as expected, the above result is consistent with (3.2).
Figures 1 and 2 show the αL-exponential function for some values of parameter α.

Theorem 3.3. *The αL-exponential function $\mathfrak{e}_\alpha(\lambda x)$, $\alpha\in[0,\infty)$, $x\geq 0$, $\lambda\in\mathbb{R}$, is an eigenfunction of the operators*

$$
\underbrace{\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}\cdots\frac{\mathrm{d}}{\mathrm{d}x}x\frac{\mathrm{d}}{\mathrm{d}x}}_{r \textit{ derivatives}}D^{\alpha+1-r}, \qquad \alpha\in[0,\infty),\ r=1,\dots,n, \tag{3.11}
$$

where $n=[\alpha+1]$ is the integer part of $\alpha+1$. Moreover, for $\alpha\in(-1,0)$ the αL-exponential function $\mathfrak{e}_\alpha(\lambda x)$, $x\geq 0$, $\lambda\in\mathbb{R}$, is an eigenfunction of the operator $x^{-1}D^{\alpha+1}$.

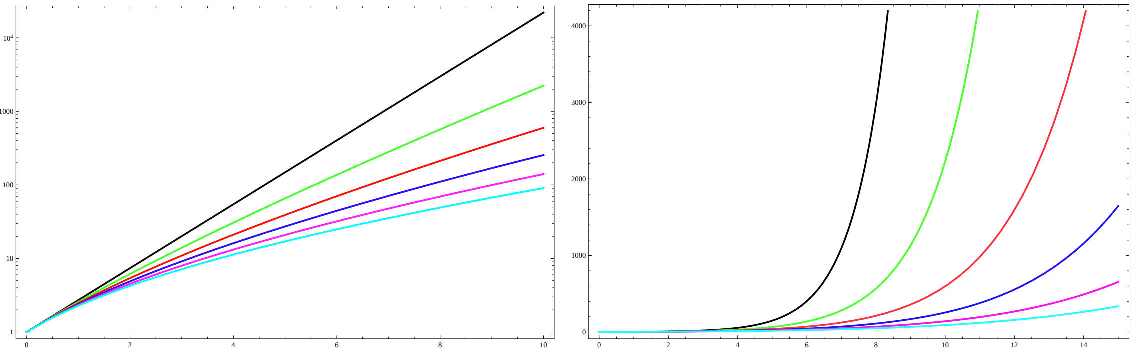

Figure 1: The αL-exponential function $\mathfrak{e}_\alpha(x)$, $x \geq 0$ in loglinear plot (left) and linear plot (right), for $\alpha = 0$ (black—classical exponential), $\alpha = 0.2$ (green), $\alpha = 0.4$ (red), $\alpha = 0.6$ (blue), $\alpha = 0.8$ (magenta), $\alpha = 1$ (cyan—0th-order Bessel–Tricomi function).

Proof. For the first part we can write that

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}}{\mathrm{d}x} \dots \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}}{\mathrm{d}x}}_{r \text{ derivatives}} D^{\alpha+1-r} \mathfrak{e}_\alpha(\lambda x) = \underbrace{\frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}}{\mathrm{d}x} \dots \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}}{\mathrm{d}x}}_{r \text{ derivatives}} \sum_{k=1}^{\infty} \lambda^k \frac{k^{\alpha+1-r} x^k}{k!^{\alpha+1}} \tag{3.12}$$

$$= \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} \lambda^k \frac{k^\alpha x^k}{k!^{\alpha+1}} = \lambda \mathfrak{e}_\alpha(\lambda x).$$

Analogously, in order to prove the second part it is sufficient to note that

$$x^{-1} D^{\alpha+1} \mathfrak{e}_\alpha(\lambda x) = x^{-1} \sum_{k=1}^{\infty} \lambda^k \frac{k^{\alpha+1} x^k}{k!^{\alpha+1}} = \lambda \mathfrak{e}_\alpha(\lambda x). \tag{3.13}$$

□

Remark 3.3. *An interesting specific case of Theorem 3.3 is when $r = 1$. In this case the αL-exponential function is an eigenfunction of the operator $\frac{\mathrm{d}}{\mathrm{d}x} D^\alpha = \mathfrak{D}^\alpha$ (see Definition 2.4). This can be explained also by recalling that*

$$\underbrace{\frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}}{\mathrm{d}x} \dots \frac{\mathrm{d}}{\mathrm{d}x} x \frac{\mathrm{d}}{\mathrm{d}x}}_{r \text{ derivatives}} D^{\alpha+1-r} = \frac{\mathrm{d}}{\mathrm{d}x} \delta^{r-1} D^{\alpha+1-r} = \frac{\mathrm{d}}{\mathrm{d}x} \delta^{r-1} \delta^{n-r+1} J^{n-\alpha} = \frac{\mathrm{d}}{\mathrm{d}x} D^\alpha = \mathfrak{D}^\alpha. \tag{3.14}$$

In order to describe some applications we first recall the following result [4, page 490, Theorem 5.1] specialized for $n = 1$.

Theorem 3.4. *Consider the problem*

$$\begin{cases} D_{L,x}\, S(x,t) = \frac{\partial}{\partial t} S(x,t), & x > 0, t \geq 0, \\ S(0,t) = s(t), \end{cases} \tag{3.15}$$

with the analytic boundary condition $s(t)$. The operational solution is given by

$$S(x,t) = e_1 \left(x \frac{\partial}{\partial t} \right) s(t) = \sum_{k=0}^{\infty} \frac{x^k}{k!^2} \frac{\partial^k}{\partial t^k} s(t). \tag{3.16}$$

The operational solution (3.16) becomes an effective solution when the series converges, and this depends upon the actual form of the boundary condition $S(0,t)$. Notice that, from a physical point of view, this *Laguerre-heat equation* is in fact a diffusion equation with a space-dependent diffusion coefficient.
In a similar way we can solve more general fractional partial differential equations. We notice that the utility of operational methods to solve fractional differential equations was pointed out by different authors, see for example Hilfer et al. [8], Luchko and Gorenflo [11], Garra and Polito [6], Tomovski et al. [15]. See also the book by Yakubovich and Luchko [16] and the references therein.
Here we present the following result for an initial value problem involving the operator $\mathfrak{D}^\alpha$.

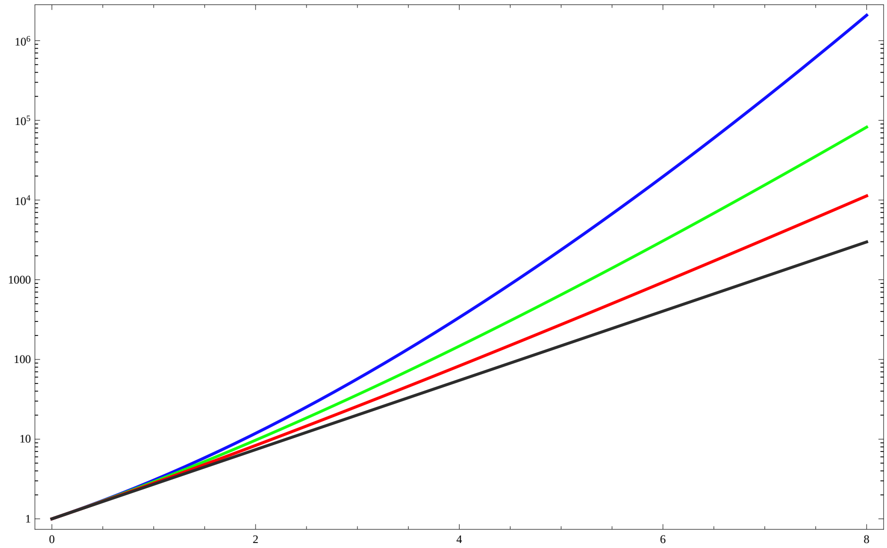

Figure 2: The αL-exponential function $\mathfrak{e}_\alpha(x)$, $x \geq 0$ in loglinear plot for $\alpha = 0$ (black—classical exponential), $\alpha = -0.1$ (red), $\alpha = -0.2$ (green), $\alpha = -0.3$ (blue).

Theorem 3.5. *Consider the following initial value problem.*

$$\begin{cases} \mathfrak{D}_t^\alpha f(x,t) = \Theta_x f(x,t), \quad t > 0, \\ f(x,0) = g(x), \end{cases} \tag{3.17}$$

for $\alpha \in (0,1)$, with an analytic initial condition $g(x)$ and where Θ_x is a generic linear integro-differential operator with constant coefficients, acting on x, and which satisfies the semigroup property, that is $\Theta_x\Theta_y = \Theta_{x+y}$. By applying Theorem 3.3 for $r = 1$, $\alpha \in (0,1)$, the operational solution to (3.17) is given by

$$f(x,t) = \mathfrak{e}_\alpha(t\,\Theta_x)\,g(x) = \sum_{r=0}^{\infty} \frac{t^r \Theta_x^r}{r!^{\alpha+1}} g(x). \tag{3.18}$$

Again, when the operational solution (3.18) converges, it becomes the effective solution.

Corollary 3.1. *Consider the following boundary value problem.*

$$\begin{cases} \mathfrak{D}_x^\alpha w(x,t) = \Xi_t w(x,t), \quad \alpha \in (0,1),\ t \geq 0,\ x > 0, \\ w(0,t) = h(t), \end{cases} \tag{3.19}$$

with an analytic boundary condition $h(t)$ and where Ξ_t is a generic linear integro-differential operator with constant coefficients, acting on t, and which satisfies the semigroup property. Analogously to Theorem 3.5, the operational solution to (3.19) is given by

$$w(x,t) = \mathfrak{e}_\alpha(x\,\Xi_t)\,h(t) = \sum_{r=0}^{\infty} \frac{x^r \Xi_t^r}{r!^{\alpha+1}} h(t). \tag{3.20}$$

It is important to remark that the operational solutions (3.18) and (3.20) are indeed effective solutions whenever the corresponding series are convergent.

Example 3.1. *Consider the following initial value problem.*

$$\begin{cases} \mathfrak{D}_t^\alpha\, u(x,t) = \frac{\partial^2}{\partial x^2} u(x,t), \quad \alpha \in (0,1),\ t > 0, \\ u(x,0) = \sin x, \end{cases} \tag{3.21}$$

By Theorem 3.5 the solution is given by

$$u(x,t) = \sin x\ \mathfrak{e}_\alpha(t) = \sin x \sum_{k=0}^{\infty} \frac{t^k}{k!^{\alpha+1}}, \tag{3.22}$$

which is clearly convergent for each $(x,t) \in \mathbb{R} \times \mathbb{R}^+$.

4 An application to a modified Lamb–Bateman equation

In [1] the authors have shown that a modified Lamb–Bateman integral equation can be written as a fractional equation involving Hadamard derivatives of order $1/2$. They have studied the equation

$$\int_0^\infty \mathrm{d}y\, u\left(e^{-y^2}x\right) = f(x), \tag{4.1}$$

where $u(x)$ is the unknown function that has to be determined and $f(x)$ is a known continuous function. Recalling that, for the dilation operator $\hat{E}(\lambda) = e^{\lambda x \frac{\mathrm{d}}{\mathrm{d}x}}$, we have

$$\hat{E}(\lambda)\, u(x) = e^{\lambda x \frac{\mathrm{d}}{\mathrm{d}x}} u(x) = u(e^{\lambda} x), \tag{4.2}$$

then equation (4.1) can consequently be written as

$$\int_0^\infty \mathrm{d}y\, e^{-y^2 x \frac{\mathrm{d}}{\mathrm{d}x}} u(x) = f(x), \tag{4.3}$$

and therefore

$$u(x) = \frac{2}{\sqrt{\pi}} \left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^{1/2} f(x) = \frac{2}{\sqrt{\pi}} D^{1/2} f(x), \tag{4.4}$$

that is, an integral equation involving Hadamard derivatives of order $1/2$.
Note that, in the general case, the function f depends on u. The simplest choice is $f(u(x)) = u(x)$. In this specific case we have

$$u(x) = \frac{2}{\sqrt{\pi}} D^{1/2} u(x). \tag{4.5}$$

An application of the x-derivative to both sides of the above equation leads to a formulation of the Lamb–Bateman equation involving the operator $\mathfrak{D}^\alpha$.

$$\mathfrak{D}^{1/2} u(x) = \frac{\sqrt{\pi}}{2} \frac{\mathrm{d}}{\mathrm{d}x} u(x). \tag{4.6}$$

By using a simple *ansatz*, that is $u(x) = x^\beta$, where β is to be specified, we find by substitution

$$\beta^{3/2} x^{\beta-1} = \frac{\sqrt{\pi}}{2} \beta x^{\beta-1}, \tag{4.7}$$

so that $u(x) = x^\beta$ is a simple exact solution if $\beta = \pi/4$.
In general we can consider the equation

$$\int_0^\infty \mathrm{d}y\, u\left(e^{-y^{1/\mu}}x\right) = f(x), \qquad \mu \in (0,\infty), \tag{4.8}$$

which can be rewritten, as before, as

$$\int_0^\infty \mathrm{d}y\, e^{-y^{1/\mu} x \frac{\mathrm{d}}{\mathrm{d}x}} u(x) = f(x). \tag{4.9}$$

By evaluating the integral we then have that

$$u(x) = \frac{1}{\Gamma(\mu+1)} \left(x \frac{\mathrm{d}}{\mathrm{d}x}\right)^{\mu} f(x) = \frac{1}{\Gamma(\mu+1)} D^{\mu} f(x). \tag{4.10}$$

Again, the simplest choice of $f(u(x)) = u(x)$ leads to

$$u(x) = \frac{1}{\Gamma(\mu+1)} D^{\mu} u(x), \qquad \mu \in (0,\infty). \tag{4.11}$$

The related equation involving the operator $\mathfrak{D}^\alpha$ is

$$\mathfrak{D}^{\mu} u(x) = \Gamma(\mu+1) \frac{\mathrm{d}}{\mathrm{d}x} u(x), \tag{4.12}$$

and by choosing again $u(x) = x^\beta$ we obtain that $\beta = [\Gamma(\mu+1)]^{1/\mu}$.

References

[1] Babusci, D., Dattoli, G. and Sacchetti, D. Integral equations, fractional calculus and shift operators. *arXiv:1007.5211v1 [math-ph]*, 2010.

[2] Babusci, D., Dattoli, G. and Sacchetti, D. The Lamb–Bateman integral equation and the fractional derivatives. *Fractional Calculus And Applied Analysis*, 14(2):317–320, 2011.

[3] Dattoli, G., He, M.X. and Ricci, P.E. Eigenfunctions of Laguerre-Type Operators and Generalized Evolution Problems. *Mathematical and Computer Modelling*, 42:1263–1268, 2005.

[4] Dattoli, G. and Ricci, P.E. Laguerre-type exponentials and the relevant L-circular and L-hyperbolic functions. *Georgian Mathematical Journal*, 10:481–494, 2003.

[5] Doetsch, G. *Introduction to the Theory and Application of the Laplace Transformation.* Springer, Berlin, 1974.

[6] Garra, R. and Polito, F. Analytic solutions of fractional differential equations by operational methods. *Applied Mathematics and Computation*, 218:10642–10646, 2012.

[7] Hadamard, J. Essai sur l'étude des fonctions données par leur développment de Taylor. *J. Math. Pures et Appl.*, 8:101–182, 1892.

[8] Hilfer, R., Luchko, Y. and Tomovski, Ž., Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives. *Fractional Calculus and Applied Analysis*, 12(3):299–318, 2009.

[9] Kilbas, A. Hadamard-type fractional calculus. *J. Korean. Math. Soc.*, 38:1191–1204, 2001.

[10] Kiryakova, V. The multi-index Mittag-Leffler functions as an important class of special functions of fractional calculus. *Computers and Mathematics with Applications*, 59(5):1885–1895, 2010.

[11] Luchko, Y.F. and Gorenflo, R. An operational method for solving fractional differential equations. *Acta Mathematica Vietnamica*, 24:207–234, 1999.

[12] Podlubny, I. *Fractional Differential Equations.* Academic Press, San Diego, 1999.

[13] Ricci, P.E. and Tavkhelidze, I. An introduction to operational techniques and special polynomials. *Journal of Mathematical Sciences*, 157(1):161–189, 2009.

[14] Samko, S.G., Kilbas, A.A. and Marichev, O.I. *Fractional Integrals and Derivatives: Theory and Applications.* Gordon and Breach, 1993.

[15] Tomovski, Ž., Hilfer, R. and Srivastava, H.M., Fractional and operational calculus with generalized fractional derivative operators and Mittag-Leffler type functions. *Integral Transforms and Special Functions*, 21(11):797–814, 2010.

[16] Yakubovich, S.B. and Luchko, Y.F. *The Hypergeometric Approach to Integral Transforms and Convolutions.* Kluwer Academic Publishers, 1994.